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# Bicritical dynamic scaling and dynamic crossover in the bond-diluted Glauber Ising chain

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**Abstract.** We present an investigation of the dynamic critical behaviour near the percolation bicritical point of the 1D bond-diluted Glauber Ising model. The expected bicritical dynamic scaling form is rigorously established for the decay functions describing the decay of the equilibrium wavevector-dependent correlation function and the decay of the magnetisation from a (non-equilibrium) wavevector-dependent initial state. The corresponding characteristic decay times are evaluated exactly and exhibit a crossover from 'pure' to percolation dominated behaviour on increasing the dilution. The results in the percolation regime are in agreement with recent calculations in which domain boundary diffusion methods are applied to a wider class of kinetic Ising models.

## 1. Introduction

The dynamic scaling hypothesis which was first proposed by Ferrell *et al* (1967) and subsequently generalised by Halperin and Hohenberg (1967), furnishes a description of the low frequency and long-wavelength dynamics of a system near a continuous phase transition. According to the hypothesis the lifetime of the  $k$ th Fourier component of the equilibrium-order parameter density correlation function  $C(k, t)$  for a pure system with relaxational dynamics is assumed to obey the dynamic scaling relation

$$\tau(k) = \xi^z f(k\xi) \quad (1)$$

where  $\xi$  is the thermal correlation length and  $z$  is the dynamic exponent. Defining a normalised decay function

$$g(t) = C(k, t)/C(k, 0) \quad (2)$$

the dynamic scaling hypothesis further leads to the following scaling form for  $g(t)$ , which incorporates (1):

$$g(k, \xi, t) = G(t/\xi^z, k\xi). \quad (3)$$

The  $k$  and  $\xi$  dependences of  $g$  have been put in explicitly in the left-hand side of (3). The results (1) and (3) are expected to be valid at criticality, where  $k \rightarrow 0$ ,  $\xi \rightarrow \infty$ , but  $k\xi$  is finite and arbitrary. The original dynamic scaling hypothesis describing the decay of order parameter fluctuations in the equilibrium ensemble at criticality has been

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extended to other densities of the system, such as the energy density (Halperin and Hohenberg 1969), and to the decay of the order parameter density from a non-equilibrium initial state (Racz 1976, Binder 1979).

For pure systems with various types of dynamics, scaling forms such as (1) and (3) have been tested experimentally and dynamic universality classes established (Hohenberg and Halperin 1977). The picture is much less well developed for quenched diluted systems at low temperatures near the percolation threshold, which can be regarded as a bicritical point (Stauffer 1975). Suitable parameters describing the critical region in the vicinity of this point are the wavevector  $k$ , the percolation correlation length  $\xi_p$ , and the thermal correlation length of the corresponding pure one-dimensional system  $\xi_T$  (Birgeneau *et al* 1976). At criticality  $k \rightarrow 0$ ,  $\xi_p, \xi_T \rightarrow \infty$  but with the scaled variables  $k(\xi_T)^{\nu_T}$ ,  $(\xi_T)^{\nu_T}/\xi_p$  remaining finite and arbitrary.  $\nu_T$  is a static thermal exponent. Dynamic scaling ideas lead to the following generalisations of (1) and (3) to the percolation bicritical dynamics of a diluted system:

$$\tau(k) = \xi^z f(k\xi, \xi/\xi_p) \quad (4)$$

$$g(k, \xi_T, \xi_p, t) = G(t/\xi^z, k\xi, \xi/\xi_p) \quad (5)$$

where

$$\xi = (\xi_T)^{\nu_T}.$$

As  $\xi/\xi_p$  is increased from zero to infinity the system crosses over from a thermal regime, in which thermal fluctuations dominate the critical behaviour, to a percolation regime, in which the spatial decay of correlations is controlled by the percolation geometry. This thermal to percolation crossover produces a corresponding dynamic crossover in the functions  $g(t)$  and  $\tau(k)$ . In this paper, the scaling form (5), and hence (4), is established rigorously for the equilibrium wavevector-dependent spin-spin correlation function decay, and the decay of the wavevector-dependent magnetisation from a non-equilibrium initial state, in an exactly soluble one-dimensional system—the bond-diluted Glauber Ising model which has a percolation bicritical point at  $p=1$ , corresponding to all bonds present, and  $T=0$ . The thermal to percolation crossover can then be regarded as a crossover from a pure to a percolation regime. The exponent  $\nu_T$  is unity for this model and the value of the dynamic exponent  $z$  is 2. Explicit expressions are obtained for the lifetimes  $\tau(k)$  at criticality and an interpretation of the pure to percolation crossover is given in terms of domain boundary diffusion arguments. The layout of the remainder of the paper is as follows. In § 2 single spin flip Glauber dynamics is developed for the disordered Ising chain and specialisation to the case of random bond dilution is then made. Formal expansions for the Laplace transforms of the equilibrium correlation function decay and non-equilibrium magnetisation decay in terms of configuration-averaged finite chain Green functions are written down. A continuum version of the general theory appropriate at criticality is then used to establish the dynamic scaling form for the decay functions  $g(t)$  in § 3, and the corresponding lifetimes  $\tau(k)$  are evaluated explicitly in § 4. A discussion of the results and interpretation in terms of domain boundary diffusion arguments is given in § 5.

## 2. General theory

The continuing popularity of kinetic Ising models in the literature is due to the fact that these idealised dynamical models nevertheless provide a useful description of the

dynamics in a variety of real systems. (For a list of applications see Kawasaki 1972.) An additional attractive feature of the models is their suitability for computer simulation experiments (see e.g. Binder 1976, 1979). Furthermore, the pure one-dimensional Ising system with a special type of single spin flip dynamics has been solved exactly (Glauber 1963, Bedeaux *et al* 1970) and dynamic critical exponents have been obtained, and universality classes established, for pure Ising chains with more general single spin flip and double spin flip dynamics (Haake and Thol 1980, Cordery *et al* 1981).

In the single spin flip dynamics initiated for the Ising chain by Glauber (1963), all the information about the chain at time  $t$  is contained in the probability function  $P(\sigma_1 \dots \sigma_N, t)$  which is the probability that the spin configuration of the chain is  $\{\sigma_1 \dots \sigma_n\}$  at time  $t$ , and whose evolution is governed by the master equation

$$\dot{P}(\sigma_1 \dots \sigma_N, t) = - \left( \sum_j W_j(\sigma_j) \right) P(\sigma_1 \dots \sigma_N) + \sum_j W_j(-\sigma_j) P(\sigma_1 \dots -\sigma_j \dots \sigma_N, t); \quad (6)$$

$W_j(\sigma_j)$  is the transition rate for the process  $\{\sigma_1 \dots \sigma_j \dots \sigma_N\} \rightarrow \{\sigma_1 \dots -\sigma_j \dots \sigma_N\}$  and, for a chain with nearest-neighbour interactions, the spin dependence of  $W_j(\sigma_j)$  is assumed to be a function only of the orientation of  $\sigma_j$  relative to its nearest neighbours  $\sigma_{j-1}$  and  $\sigma_{j+1}$ . The following equations of motion for the  $n$ -spin equal time correlation functions

$$q_{l(1) \dots l(n)}(t) = \langle \sigma_{l(1)}(t) \sigma_{l(2)}(t) \dots \sigma_{l(n)}(t) \rangle$$

may easily be obtained from (6) (Glauber 1963, Bedeaux *et al* 1970):

$$\dot{q}_{l(1) \dots l(n)}(t) = -2 \sum_{i=1}^n \langle \sigma_{l(1)}(t) \dots \sigma_{l(n)}(t) W_{l(i)}[\sigma_{l(i)}(t)] \rangle \quad (7)$$

where  $\langle \dots \rangle = \sum_{\{\sigma\}} P(\{\sigma\}, t) \dots$

For the general nearest-neighbour bond-disordered Ising chain, with Hamiltonian

$$-\beta \mathcal{H} = \sum_i K_i \sigma_i \sigma_{i+1}, \quad \sigma_i = \pm 1 \quad (8)$$

the most general form for  $W_j(\sigma_j)$  satisfying the requirements of detailed balance and spin dependence only on the orientation of  $\sigma_j$  relative to its nearest neighbours is the following

$$W_j(\sigma_j) = \frac{1}{2} \Gamma_j (1 + \delta_j \sigma_{j-1} \sigma_{j+1}) \{ 1 - \frac{1}{2} (\gamma_j^+ \sigma_{j-1} + \gamma_j^- \sigma_{j+1}) \sigma_j \}$$

with

$$\gamma_j^\pm = \tanh(K_{j-1} + K_j) \pm \tanh(K_{j-1} - K_j). \quad (9)$$

$\Gamma_j$  and  $\delta_j$  are arbitrary, except that  $|\delta_j| < 1$ ,  $\Gamma_j > 0$  to ensure a positive  $W_j(\sigma_j)$ . The specialisation  $\delta_j = 0$  for all  $j$  in (9), made by Glauber in the pure model, leads to the important simplification of closed systems of equations for the  $n$ -spin equal time correlation functions in (7) in terms of correlations of  $n$  and fewer spins. We further follow previous authors in making the specialisation  $\Gamma_j = \Gamma$  in (9) and absorbing the constant  $\Gamma$  into the definition of the timescale.

Next we specialise to the bond-diluted Ising chain, remarking that the site-diluted case is related in a trivial way for a chain (see e.g. Wortis 1974). Diluting a one-dimensional system breaks it up into finite segments, and if bonds are removed at

random with probability  $q$ , so that the  $K_i$  in (8) are distributed according to

$$P(K_i) = (1 - q)\delta(K_i - K) + q\delta(K_i),$$

it is easy to see that the probability that a randomly chosen site belongs to an  $n$ -bond segment is given by

$$P(n) = (n + 1)q^2(1 - q)^n. \tag{10}$$

The percolation correlation function is the probability that a pair of sites separated by  $r$  lies in the same segment and a simple calculation yields the following result for the percolation correlation length

$$\xi_p = [-\ln(1 - q)]^{-1}. \tag{11}$$

The correlation function for Ising spins on a bond-diluted chain at finite temperature may also easily be calculated. The result is

$$C(r) = \exp(-r/\xi)$$

with

$$\xi^{-1} = \xi_T^{-1} + \xi_p^{-1}. \tag{12}$$

$\xi_T = [-\ln(\tanh 2K)]^{-1}$  is the pure 1D Ising thermal correlation length.

Turning now to Glauber dynamics on an  $n$ -bond chain segment, with sites labelled from 0 to  $n$ , we obtain from (9) with  $\delta_j = 0$  and  $\Gamma_j = \Gamma$  absorbed into the timescale, the rates

$$\begin{aligned} W[\sigma_l] &= \frac{1}{2}[1 - \frac{1}{2}\gamma\sigma_l(\sigma_{l-1} + \sigma_{l+1})], & l \neq 0, n \\ W[\sigma_0] &= \frac{1}{2}(1 - \eta\sigma_0\sigma_1), & W[\sigma_n] = \frac{1}{2}(1 - \eta\sigma_n\sigma_{n-1}) \end{aligned} \tag{13}$$

$$\text{where } \gamma = \tanh 2K, \quad \eta = \tanh K.$$

Substituting (13) into (7), and taking the Laplace transform, we obtain the following equation of motion for the Laplace-transformed single-site magnetisation  $\tilde{q}_i(s)$

$$\tilde{q}_i(s) = \sum_{l'=0}^n \tilde{G}_{il'}^{(n)}(s)q_{l'}(0) \tag{14}$$

where the  $n$ -bond segment Green function  $\tilde{G}_{il'}^{(n)}(s)$  satisfies the equation

$$(s + 1 - \gamma)\tilde{G}_{il'}^{(n)}(s) - \frac{1}{2}\gamma(\tilde{G}_{i+1l'}^{(n)}(s) + \tilde{G}_{i-1l'}^{(n)}(s) - 2\tilde{G}_{il'}^{(n)}(s)) = \delta_{il'}, \quad 1 \leq l \leq n - 1 \tag{15}$$

subject to the boundary conditions

$$\begin{aligned} (s + 1 - \eta)\tilde{G}_{0l'}^{(n)}(s) - \eta(\tilde{G}_{1l'}^{(n)}(s) - \tilde{G}_{0l'}^{(n)}(s)) &= \delta_{0l'} \\ (s + 1 - \eta)\tilde{G}_{nl'}^{(n)}(s) - \eta(\tilde{G}_{n-1l'}^{(n)}(s) - \tilde{G}_{nl'}^{(n)}(s)) &= \delta_{nl'}. \end{aligned} \tag{16}$$

$\tilde{G}_{il'}^{(n)}(s)$  is zero if  $l$  or  $l'$  are outside the range 0 to  $n$ . We consider now the  $i$ th site of a bond-diluted Ising chain with ensemble-averaged initial condition  $q_i(0)$  for each bond configuration. For those bond configurations in which  $i$  is the  $l$ th site of an  $n$ -bond segment, we obtain from (14),

$$\tilde{q}_i(s) = \sum_{l'=0}^n \tilde{G}_{il'}^{(n)}(s)q_{i-l+l'}(0). \tag{17}$$

Configuration averaging (17) over all bond configurations yields the following:

$$\overline{\tilde{q}_i(s)} = \sum_{n=0}^{\infty} P(n) \frac{1}{n+1} \sum_{l=0}^n \sum_{l'=0}^n \tilde{G}_{ll'}^{(n)}(s) q_{i-l+l'}(0). \tag{18}$$

$P(n)$  is given by (10). Defining the Fourier transform

$$\tilde{q}(k, s) = \sum_j \exp(-ikj) \overline{\tilde{q}_i(s)},$$

we obtain from (18), using (10),

$$\tilde{q}(k, s) = \sum_{n=0}^{\infty} q^2(1-q)^n \sum_{l=0}^n \sum_{l'=0}^n \tilde{G}_{ll'}^{(n)}(s) \exp[-ik(l-l')] q(k, 0).$$

Then the Laplace transform of the decay function for the non-equilibrium  $k$ -dependent magnetisation decay  $g_m(t) = q(k, t)/q(k, 0)$  is given by

$$\tilde{g}_m(s) = \sum_{n=0}^{\infty} q^2(1-q)^n \sum_{l=0}^n \sum_{l'=0}^n \tilde{G}_{ll'}^{(n)}(s) \exp[-ik(l-l')]. \tag{19}$$

We next derive a corresponding result for the decay function of the wavevector-dependent time-delayed two-spin correlation function in equilibrium. The general two-spin time-dependent correlation function is given by

$$C_{ll'}(t', t) = \langle \sigma_l(t+t') \sigma_{l'}(t') \rangle$$

where

$$\langle \dots \rangle = \sum_{\{\sigma, \sigma'\}} P(\{\sigma(t+t'), \sigma'(t')\}, t') \dots \tag{20}$$

$P(\{\sigma(t+t'), \sigma'(t')\}, t')$  in (20) is the probability that the spin configuration of the system is  $\{\sigma'\}$  at time  $t'$  and  $\{\sigma\}$  at time  $t+t'$ . In equilibrium  $C_{ll'}(t', t)$  is independent of  $t'$  and is abbreviated to  $C_{ll'}(t)$ . We show in appendix 1 that the result corresponding to (14) for  $C_{ll'}(s)$  on an  $n$ -bond segment is

$$\begin{aligned} \tilde{C}_{ll'}(s) &= \sum_{l''=0}^n \tilde{G}_{ll''}^{(n)}(s) \eta^{|l'-l''|}, & 0 \leq l, l' \leq n \\ &= 0 & \text{otherwise.} \end{aligned} \tag{21}$$

Using (21) the following result for the configuration average of  $\tilde{C}_{ij}(s)$  on the bond-disordered chain may easily be obtained (cf derivation of (17)):

$$\overline{\tilde{C}_p(s)} = \sum_{n=|p|}^{\infty} (1-q)^n q^2 \sum_{l=0}^{n-|p|} \sum_{l'=0}^n \tilde{G}_{ll'}^{(n)}(s) \eta^{|l'-|p|-l|}$$

with

$$p = i - j. \tag{22}$$

We notice that configuration averaging restores the translational invariance of the bond-disordered correlation function. Taking the Fourier transform

$$\tilde{C}(k, s) = \sum_p \exp(-ikp) \overline{\tilde{C}_p(s)}$$

of (22) and rearranging yields the result

$$\begin{aligned} \tilde{C}(k, s) &= 2 \sum_{p=0}^{\infty} \cos(kp) \sum_{n=p}^{\infty} (1-q)^n q^2 \sum_{l=0}^{n-p} \sum_{l'=0}^n G_{ll'}^{(n)}(s) \eta^{|l'-p-l|} \\ &= \tilde{g}_c(s) C(k, 0) \end{aligned} \tag{23}$$

where  $\tilde{g}_c(s)$  is the Laplace transform of the decay function defined in (2). For non-exponential decays  $g(t)$  the definition of a lifetime is somewhat arbitrary. We shall use the simple and widely used definition

$$\tau = \int_0^{\infty} g(t) dt. \tag{24}$$

Then

$$\tau_i = \tilde{g}_i(0), \quad i = m, c. \tag{25}$$

Equations (15) and (16) may readily be solved to obtain the segment Green function  $\tilde{G}_{ll'}^{(n)}(s)$ . On substituting the result into the expressions for  $\tilde{g}_i(s)$  ((19) and (23) for  $i = m, c$  respectively) we find that the sums over  $l, l'$  and  $p$  may be evaluated explicitly and we can thus obtain the exact results, valid at arbitrary dilution and temperature, for  $\tilde{g}_i(s)$ , and hence for the lifetimes  $\tau_i$  through (25), in terms of a sum over  $n$ . This procedure is not carried out here, since we are interested in the critical limit  $q \rightarrow 0, T \rightarrow 0, k \rightarrow 0$ , where a simpler continuum formulation of the theory of the present section is appropriate. This leads directly to the scaling form (5) for the decay functions  $g_i(t)$  and to closed-form expressions for the characteristic times  $\tau_i$ .

### 3. Criticality and the continuum approach

At criticality, the three relevant lengths in the system,  $\xi_p, \xi_T$  and the wavelength  $\lambda$  all diverge, while the unit lattice spacing becomes irrelevant. The scaled variables  $\tilde{k} = k\xi_T$  and  $\tilde{q} = \xi_T/\xi_p$  remain finite and arbitrary at criticality. We define the following scaled variables

$$\begin{aligned} y &= l/\xi_T, & x &= p/\xi_T, & L &= n/\xi_T \\ u &= t/\xi_T^2, & \rho &= s\xi_T^2. \end{aligned} \tag{26}$$

We further define a magnetisation density

$$q(y, u) = \xi_T q_l(t).$$

Then

$$\tilde{q}(y, \rho) = \xi_T^{-1} \tilde{q}_l(s). \tag{27}$$

Making these changes of variable in (15) and (16), letting  $\xi_T \rightarrow \infty$ , and using the results

$$\lim_{\xi_T \rightarrow \infty} (1 - \gamma) = 1/2\xi_T^2, \quad \lim_{\xi_T \rightarrow \infty} (1 - \eta) = 1/\xi_T$$

we obtain

$$(2\rho + 1)\tilde{G}^{(L)}(y, y'; \rho) - (\partial^2/\partial y^2)\tilde{G}^{(L)}(y, y'; \rho) = 2\delta(y - y'), \quad 0 < y, y' < L \tag{28}$$

with

$$\begin{aligned} \tilde{G}^{(L)}(0, y'; \rho) - (\partial/\partial y)\tilde{G}^{(L)}(0, y'; \rho) &= 0 \\ \tilde{G}^{(L)}(L, y'; \rho) + (\partial/\partial y)\tilde{G}^{(L)}(L, y'; \rho) &= 0 \end{aligned} \tag{29}$$

and

$$\tilde{G}^{(L)}(y, y'; \rho) = \xi_T^{-1} \tilde{G}_{ii'}^{(n)}(s) \quad (\text{cf } 27). \tag{30}$$

Similarly using (30), and the result that  $\lim_{q \rightarrow 0} \xi_p = 1/q$  from (11) equations (19) and (23) become

$$\tilde{g}_m(\rho) = \tilde{q}^2 \int_0^\infty dL \exp(-\tilde{q}L) \int_0^L dy \int_0^L dy' \{ \tilde{G}^{(L)}(y, y'; \rho) \exp[-i\tilde{k}(y-y')] \} \tag{31}$$

and

$$\begin{aligned} \tilde{g}_c(\rho) = \frac{2}{C(k, 0)} \xi_T \tilde{q}^2 \int_0^\infty dx \cos \tilde{k}x \int_x^\infty dL \exp(-\tilde{q}L) \\ \times \left( \int_0^{L-x} dy \int_0^L dy' \tilde{G}^{(L)}(y, y'; \rho) \exp(-|y'-x-y|) \right) \end{aligned} \tag{32}$$

where

$$\tilde{g}_i(\rho) = \xi_T^2 \tilde{g}_i(s), \quad i = m, c. \tag{33}$$

The static correlation function  $C(k, 0)$  may be evaluated using (12). The result in the critical limit is

$$C(k, 0) = 2\xi_T(1 + \tilde{q}) / [(1 + \tilde{q})^2 + \tilde{k}^2] \tag{34}$$

thus cancelling the diverging factor  $\xi_T$  in (32). Taking the inverse transform of (33), and using (34) in (32), we have that

$$g_i(t) = G_i(u, \tilde{k}, \tilde{q}) \quad \text{only for } i = m, c,$$

and is therefore of the required scaling form (5) with  $\nu_T = 1$  and  $z = 2$ . The scaling form (4) for the lifetime  $\tau_i$  is clearly obtained using (24). These times are evaluated explicitly in the next section.

#### 4. Evaluation of characteristic times

The characteristic times  $\tau_i$  are given by

$$\tau_i = \xi_T^2 \tilde{g}_i(\rho = 0), \quad i = c, m \tag{35}$$

from (25) and (33) and are evaluated by solving (28), (29) to obtain  $\tilde{G}^{(L)}(y, y'; \rho)$ , substituting into (31) and (32), and performing the integrations explicitly for  $\rho = 0$ . The solution of (28) subject to the boundary conditions (29) is easily constructed. The general solution of (28) consists of a particular integral which is the pure chain Green function plus a complementary function with two arbitrary constants. These are fixed by substitution into (29) yielding the result

$$\begin{aligned} \tilde{G}^{(L)}(y, y'; \rho) = \nu^{-1} \{ \exp(L\nu) - \phi^2 \exp(-L\nu) \}^{-1} [ \exp[(L - |y - y'|)\nu] \\ + \phi^2 \exp[-(L - |y - y'|)\nu] + 2\phi \cosh(L - y - y')\nu ] \end{aligned}$$



where

$$\nu = (2\rho + 1)^{1/2}, \quad \phi = (\nu - 1)/(\nu + 1). \quad (36)$$

Setting  $\rho = 0$  in (36) and substituting into (31) and (32), using (34) and (35), yields the following expressions for the characteristic times  $\tau_m, \tau_c$ :

$$\tau_m = \xi_T^2 \tilde{q}^2 \int_0^\infty dL \exp(-\tilde{q}L) \int_0^L dy \int_0^L dy' \exp[-(|y - y'| + i\tilde{k}(y - y'))] \quad (37)$$

$$\begin{aligned} \tau_c = \xi_T^2 \frac{\tilde{q}^2 \{(1 + \tilde{q})^2 + \tilde{k}^2\}}{(1 + \tilde{q})} \int_0^\infty dx \cos \tilde{k}x \int_0^\infty dL \exp(-\tilde{q}L) \\ \times \left( \int_0^{L-x} dy \int_0^L dy' \exp[-(|y - y'| + |y' - x - y|)] \right). \end{aligned} \quad (38)$$

On carrying out the elementary integrations in (37) and (38) we obtain the results

$$\tau_m = 2\xi_T^2 (\tilde{q} + 1) / [(\tilde{q} + 1)^2 + \tilde{k}^2] \quad (39)$$

$$\tau_c = \xi_T^2 [(3\tilde{q} + 4)(\tilde{q} + 1)^2 + \tilde{k}^2 \tilde{q}] / \{(\tilde{q} + 1)^2 + \tilde{k}^2\} (\tilde{q} + 2)(\tilde{q} + 1). \quad (40)$$

The calculation leading to (39), (40) is carried out in appendix 2. An interpretation of the above results for the lifetimes  $\tau_m$  and  $\tau_c$  is given in § 5.

## 5. Interpretation and discussion

The explicit results, valid at criticality, for  $\tau_m$  and  $\tau_c$  in (39) and (40) above allow us to study the effect on the dynamics produced by the pure to percolation crossover as  $\tilde{q} = \xi_T / \xi_p$  is increased from zero to infinity. We first note that on setting  $\tilde{q}$  to zero in (39) and (40) we recover the pure chain result

$$\tau_m = \tau_c = 2\xi_T^2 / (1 + \tilde{k}^2) \quad (41)$$

which can be obtained from the original Glauber (1963) solution. Letting  $\tilde{q} \rightarrow \infty$  on the other hand we obtain

$$\tau_m = 2\xi_T^2 \tilde{q} / (\tilde{q}^2 + \tilde{k}^2), \quad \tau_c = \xi_T^2 (3\tilde{q}^2 + \tilde{k}^2) / \tilde{q}(\tilde{q}^2 + \tilde{k}^2). \quad (42)$$

When  $\tilde{k} = 0$ , the result (41) (up to a numerical factor) and (42) (exactly) can be derived using domain boundary diffusion arguments. This approach was initiated by Cordery *et al* (1981) who argued for the pure chain that the decay of correlations at low temperatures is dominated by the diffusion of domain boundaries. A measure of the correlation function decay time is the time  $\tau$  taken for an average domain, length  $2\xi_T$ , to be destroyed by diffusion of the boundaries. The diffusion rate for a boundary to move a single step is of order unity (from the first of equations (13)) and therefore  $\tau \propto \xi_T^2$  from simple random walk arguments. The  $\xi_T$  dependence of the magnetisation decay from a uniformly magnetised initial state, on the other hand, arises from a subtle balance of domain boundary diffusion and domain pair creation processes. The domain boundary diffusion approach can be extended to the percolation critical regime  $\tilde{q} \rightarrow \infty$  (Harris 1984) where at any instant the spins in the vast majority of segments are completely ordered. The dominant process producing segment flip and hence destroying spin correlations/initial magnetisation in the chain is found to be the flip of an end

spin followed by diffusion of the resulting domain boundary to the other end of the segment where it is destroyed, thus having reversed the magnetisation of the segment. The rate of end spin flip (two ends) is  $1/\xi_T$  from (13) while the rate of success for diffusion of a domain boundary initially at an end bond to the other end, where it is destroyed, is proportional to  $1/\xi_p$  for a segment of length  $\xi_p$  (Harris 1984). Then the average time  $\tau$  taken for the reversal of the magnetisation of a typical segment is given by

$$\tau \propto \xi_T \xi_p = \xi_T^2 / \tilde{q}$$

in agreement, apart from factors of order unity, with the exact results obtained by setting  $\tilde{k}$  to zero in (42). A more careful calculation based on the above arguments (Harris 1984) reproduces the exact  $\tilde{k} = 0$  results for  $\tau_m$  and  $\tau_c$ .

## 6. Conclusions

In conclusion, an analytic treatment of Glauber dynamics for the bond-diluted Ising chain has been developed and a continuum approximation appropriate at criticality was used to establish the expected bicritical dynamic scaling form for the wavevector-dependent magnetisation and equilibrium correlation function decay functions. The corresponding lifetimes were evaluated and are found to be in agreement with the predictions of calculations based on the diffusion of domain boundaries. The dynamic crossover which occurs as the statics crosses over from the pure to the percolation regime is identified as a crossover from pure diffusion of existing domain boundaries, and for the magnetisation decay, bulk spin flip creating a pair of domain boundaries, to boundary creation at the ends of segments by end spin flips, and the subsequent reversal of the segment spin by the diffusion of these boundaries across the segment.

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## Appendix 1. Correlation function decay in a bond-disordered chain

In general, the time-dependent single spin magnetisation on a single configuration of a bond-disordered chain, with Hamiltonian (8), can be written in the form

$$q_i(t) = \sum_j G_{ij}(t) q_j(0) \quad (\text{A1.1})$$

where  $G_{ij}(t)$  is the Green function for the particular bond configuration of the chain. We now show that the following corresponding form holds for the correlation function  $C_{ik}(t', t)$  defined in (20):

$$C_{ik}(t', t) = \sum_j G_{ij}(t) q_{jk}(t'). \quad (\text{A1.2})$$

$q_{ik}(t')$  is the two spin equal time correlation function defined in (7). Clearly,

$$C_{ik}(t', t) = \sum_{\{\sigma'\}} P(\{\sigma'\}, t') \sigma'_k(t') \left[ \sum_{\{\sigma\}} P(\{\sigma|\sigma'\}, t') \sigma_i(t+t') \right] \tag{A1.3}$$

where  $P(\{\sigma|\sigma'\}, t')$  = probability that spin configuration is  $\{\sigma\}$  at time  $t+t'$  given  $\{\sigma'\}$  at time  $t'$ . Using (A1.1) we may write

$$\sum_{\{\sigma\}} P(\{\sigma|\sigma'\}, t') \sigma_i(t+t') = \sum_j G_{ij}(t) \sigma'_j(t'). \tag{A1.4}$$

Substituting (A1.4) into (A1.3) yields the result (A1.2). In equilibrium  $q_{jk}(t')$  is independent of  $t'$ , and is just  $\eta^{|k-j|}$  for  $k$  and  $j$  both lying within the same segment of a bond-diluted chain and zero otherwise. Hence the result (21) follows.

**Appendix 2. Evaluation of characteristic times**

We carry out the evaluation of the integrals in (37) and (38) to derive the results (39) and (40). Making the transformation of variables

$$x = y - y', \quad y'' = y$$

in (37) and carrying out the integration over  $y''$  we obtain

$$\tau_m = 2\xi_T^2 \tilde{q}^2 \int_0^\infty dx (\cos \tilde{k}x) \exp(-x) \int_x^\infty dL \exp(-\tilde{q}L)(L-x). \tag{A2.1}$$

Similarly, evaluation of the integrals over  $y$  and  $y'$  in (38) yields the result

$$\begin{aligned} \tau_c = & \frac{\xi_T^2 \tilde{q}^2 [(1+\tilde{q})^2 + \tilde{k}^2]}{(1+\tilde{q})} \int_0^\infty dx \cos \tilde{k}x \\ & \times \int_x^\infty dL \exp(-\tilde{q}L) \{ [L - \frac{1}{2} + xL - x - x^2] e^{-x} + \frac{1}{2} \exp[-(2L-x)] \}. \end{aligned} \tag{A2.2}$$

The remaining integrals in (A2.1) and (A2.2) may be evaluated in a simple way by using the following easily proved result

$$\int_0^\infty dx x^m \exp(-bx) \int_x^\infty dL L^n \exp(-aL) = \frac{n!m!}{a^{n+1}(a+b)^{m+1}} \sum_{r=0}^n \frac{a^r (a+b)^{-r} (m+r)!}{n!r!} \tag{A2.3}$$

where  $m$  and  $n$  are integers  $\geq 0$  and  $\text{Re}(a+b) > 0$ . Writing  $\cos \tilde{k}x$  as  $\text{Re}(\exp[-i\tilde{k}x])$  in (A2.1) and (A2.2) and using (A2.3) we obtain

$$\tau_m = 2\xi_T^2 \text{Re}\{[(\tilde{q}+1+i\tilde{k})^{-1} + \tilde{q}(\tilde{q}+1+i\tilde{k})^{-2}] - \tilde{q}(\tilde{q}+1+i\tilde{k})^{-2}\} \tag{A2.4}$$

$$\begin{aligned} \tau_c = & \frac{\xi_T^2 [(1+\tilde{q})^2 + \tilde{k}^2]}{(1+\tilde{q})} \text{Re}\{[(\tilde{q}+1+i\tilde{k})^{-1} + \tilde{q}(\tilde{q}+1+i\tilde{k})^{-2}] \\ & - \frac{1}{2}\tilde{q}(\tilde{q}+1+i\tilde{k})^{-1} + [(\tilde{q}+1+i\tilde{k})^{-2} + 2\tilde{q}(\tilde{q}+1+i\tilde{k})^{-3}] \\ & - \tilde{q}(\tilde{q}+1+i\tilde{k})^{-2} - 2\tilde{q}(\tilde{q}+1+i\tilde{k})^{-3} + \tilde{q}^2(2+\tilde{q})^{-1}(\tilde{q}+1+i\tilde{k})^{-1}\}. \end{aligned} \tag{A2.5}$$

Simplification of the RHS of (A2.4) and (A2.5) yields the results (39) and (40) respectively.

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